THE OPACITY DUE TO COMPTON SCATTERING AT RELATIVISTIC TEMPERATURES IN A SEMIDEGENERATE ELECTRON GAS

CHAO-WEN CHIN

COLUMBIA UNIVERSITY

and

INSTITUTE FOR SPACE STUDIES
Goddard Space Flight Center, NASA
New York, New York

ABSTRACT

The Rosseland mean due to Compton scattering at temperatures ranging from $kT = .01\ mc^2$ to $kT = 0.35\ mc^2$ and for densities ranging from $\rho \simeq 10^3$ gm/cc to $\rho = 10^6$ gm/cc is calculated. In most of this semidegenerate region, Compton scattering dominates over electron conduction, bound-free transitions, and free-free transitions. The results are compared with Sampson's (1959) results in the nondegenerate case and his estimates in the semidegenerate case. Upon comparison, his estimates are smaller by approximately 10-20 % in most cases than ours.

INTRODUCTION

Early calculations of the opacity due to Compton scattering have used the Thompson formula for the cross section, which is valid only for a low temperature and a nondegenerate gas. For Stellar structure studies at high temperatures ($kT \sim mc^2$ or $T \sim 6 \times 10^9$ °K) and high densities, which occur in advanced evolutionary phases, corrections to the Thompson scattering cross section and to the final state of the electron become important. Sampson (1959) has obtained expressions for the electron scattering opacity in the nondegenerate relativistic region, taking into account the correction to the Thompson cross section at high temperatures. In this paper we shall calculate the electron scattering opacity in the semidegenerate region, in the domain in which the electron opacity is dominant. This domain is shown in Figure 1.

THEORY OF RADIATION

For completeness, we shall present here a brief derivation of the theory of radiative transfer taking into account degeneracy. Our derivation is similar to that due to Sampson (1959) for the nondegenerate case.

In the usual notation the equation of radiative transfer has the following form

$$\begin{aligned} \boldsymbol{s}\cdot\nabla I(\nu,\boldsymbol{s}) = & -\mu_a(\nu)\left[1-\exp(-h\nu/kT)\right]\left[I(\nu,\boldsymbol{s})-B(\nu,T)\right] - \\ & \int_{\Omega_2}\int_{\underset{\sim}{P}} N(\underset{\sim}{P})\,d\underset{\sim}{P}\left\{1-\left[1+\exp\left(\frac{E_2-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega_2}(\nu,\underset{\sim}{s},\theta,\underset{\sim}{P})\,I(\nu,\underset{\sim}{s})\left[1+\frac{c^2}{2h\nu_2^3}I(\nu_2,\underset{\sim}{s_2})\right]d\Omega_2 \\ & +\int_{\Omega_2}\int_{\underset{\sim}{P_2}} N(\underset{\sim}{P_2})\,d\underset{\sim}{P_2}\left\{1-\left[1+\exp\left(\frac{E-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega}(\nu_2,\underset{\sim}{s_2},-\theta,\underset{\sim}{P_2})\,\frac{\nu}{\nu_2}\frac{d\nu_2}{d\nu} \\ & I(\nu_2,\underset{\sim}{s_2})\left[1+\frac{c^2}{2h\nu^3}I(\nu,\underset{\sim}{s})\right]d\Omega_2 \quad , \qquad (1) \end{aligned}$$

where $I(\nu,\mathbf{s})$ is the intensity of radiation of frequency ν travelling in the direction of the unit vector $\mathbf{s}$, and $B(\nu,T)$ is the equilibrium radiation intensity given by

$$B(\nu,T) = \frac{2h\nu^3}{c^2}\left[\exp(h\nu/kT) - 1\right]^{-1} \quad . \qquad (2)$$

In equation (1) the subscript $\underline{2}$ refers to the final states of the electron and the photon.

In the diffusion approximation we assume local thermodynamic equilibrium, in the place where I alone appears we write

$$I(\nu,\mathbf{s}) = B(\nu,T) \quad . \qquad (3)$$

Using equation (3) we obtain

$$N(\mathbf{p}_2)\,d\mathbf{p}_2\left\{1-\left[1+\exp\left(\frac{E-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega}\,\frac{\nu}{\nu_2}\,\frac{d\nu_2}{d\nu} = N(\mathbf{p})\,d\mathbf{p}\left\{1-\left[1+\exp\left(\frac{E_2-\mu}{kT}\right)\right]^{-1}\right\}$$
$$\frac{d\sigma}{d\Omega_2}\left(\frac{\nu}{\nu_2}\right)^3\exp\frac{h\nu_2-h\nu}{kT} \quad . \qquad (4)$$

Substituting equation (4) into equation (1) we obtain

$$\mathbf{s}\cdot\nabla I(\nu,\mathbf{s}) = -\mu_a(\nu)\left[1-\exp(-h\nu/kT)\right]\left[I(\nu,\mathbf{s})-B(\nu,T)\right] -$$
$$\int_{\mathbf{p}}\int_{\Omega_2} N(\mathbf{p})\,d\mathbf{p}\left\{1-\left[1+\exp\left(\frac{E_2-\mu}{kT}\right)\right]^{-1}\right\}\frac{d\sigma}{d\Omega_2}(\nu,\mathbf{s},\theta,\mathbf{p})\left\{I(\nu,\mathbf{s})\left[1+\frac{c^2}{2h\nu_2^3}I(\nu_2,\mathbf{s})\right] -\right.$$
$$\left. I(\nu_2,\mathbf{s}_2)\left[1+\frac{c^2}{2h\nu^3}I(\nu,\mathbf{s})\right]\left(\frac{\nu}{\nu_2}\right)^3\exp\frac{h\nu_2-h\nu}{kT}\right\}d\Omega_2 \quad . \qquad (5)$$

Assuming the solution to equation (5) to be

$$I(\nu,\mathbf{s}) = B(\nu,T) - l(\nu)\,\mathbf{s}\cdot\nabla B(\nu,T) + \cdots \qquad (6)$$

we obtain

$$l(\nu) = \left\{\mu_a(\nu)\left[1-\exp(-h\nu/kT)\right] + \mu_s(\nu)\right\}^{-1} \quad , \qquad (7)$$

where
$$\mu_s(\nu) = \int_{\mathbf{p}}\int_{\Omega_2} N(\mathbf{p})\,d\mathbf{p}\left\{1-\left[1+\exp\frac{E_2-\mu}{kT}\right]^{-1}\right\}\frac{d\sigma}{d\Omega_2}\,\frac{1-\exp(-h\nu/kT)}{1-\exp(-h\nu_2/kT)}$$
$$\left[1-\frac{\nu_2\,l(\nu_2)}{\nu\,l(\nu)}\cos\theta\right]d\Omega_2 \quad , \qquad (8)$$

where θ = scattering angle of the photon.

In equation (8) we have already dropped those terms which vanish upon integration over Ω to obtain the radiation flux. We now use the following approximations

$$\frac{1 - \exp(-h\nu/kT)}{1 - \exp(-h\nu_2/kT)} \simeq 1 \tag{9}$$

and

$$\frac{\ell(\nu_2)}{\ell(\nu)} \simeq 1 \quad . \tag{10}$$

Since the main contribution to the integrals comes from photons with $h\nu \simeq$ 4kT and $\langle \nu_2 \rangle$ is not very different from ν, the approximation (9) is very good. For Compton scattering $\ell(\nu)$ changes very slowly with the frequency ν. For instance, at kT = .05 mc^2 and η = + 1, μ_s($h\nu$ = 4.125 kT)/μ_s($h\nu$ =3.208 kT) = .957. Thus the error introduced in replacing $\left[1 - \nu_2 \ell(\nu_2)\cos\theta/(\nu \ell(\nu))\right]$ by $\left[1 - \nu_2 \cos\theta\, \nu^{-1}\right]$ is of the order of a few percent.

With the above approximations, equation (8) is reduced to

$$\mu_s(\nu) = \int_P \int_{\Omega_2} N(\underset{\sim}{P})\, d\underset{\sim}{P} \left(1 - \frac{1}{1 + \exp\left(\frac{E_2 - \mu}{kT}\right)}\right) \frac{d\sigma}{d\Omega_2} \left[1 - \frac{\nu_2}{\nu}\cos\theta\right] d\Omega_2 \quad . \tag{11}$$

Frank-Kamensetskii (1962) called $(d\sigma/d\Omega_2)\left[1 - \nu_2\cos\theta/\nu\right]$ the transport cross section.

In the classical limit ν_2 is equal to ν and the differential cross section is an even function of cosθ. Hence, upon integration over Ω_2, the contribution from $\nu_2\cos\theta/\nu$ will be zero and the transport cross section is equal to the scattering cross section.

The electron distribution function is given by

$$N(\underset{\sim}{P})\, d\underset{\sim}{P} = \frac{2}{h^3} \frac{P^2}{1 + \exp\left(\frac{E - \mu}{kT}\right)}\, dP\, d\Omega_P \quad . \tag{12}$$

In the degenerate region the contribution to the density function by the creation of positrons is small. The correction due to positron creation will be shown to be small.

CALCULATIONS

To facilitate computation we introduce the following nondimensional variables

$$E' \equiv E/mc^2 - 1$$
$$T' \equiv kT/mc^2$$
$$u \equiv h\nu/kT$$
$$u_2 \equiv h\nu_2/kT \quad . \tag{13}$$

In terms of these variables equations (12) and (11) can be expressed as

$$N(\underset{\sim}{p})\,d\underset{\sim}{p} = \frac{2}{\lambda_c^3}\,\frac{(E'^2 + 2E')^{1/2}(1+E')}{1 + \exp(E'/T' - \eta)}\,dE'\,d\Omega_p \tag{14}$$

and
$$\mu_s(u, kT, \eta) = \frac{2}{\lambda_c^3}\int_{E'=0}^{\infty}\int_{\Omega_p}\int_{\Omega_2}\frac{(E'^2+2E')^{1/2}(1+E')}{1+\exp(E'/T'-\eta)}\left[1 - \frac{1}{1+\exp(E'/T'+u-u_2-\eta)}\right]$$
$$\frac{d\sigma}{d\Omega_2}\left(1 - \frac{\nu_2}{\nu}\cos\theta\right)d\Omega_2\,d\Omega_p\,dE' \quad , \tag{15}$$

where λ_c is the Compton wavelength $= h/mc$. In obtaining equation (15) we have eliminated E_2 by means of the energy conservation equation $E + h\nu = E_2 + h\nu_2$.

The direction of the incident electron is chosen as the positive z-axis. The direction of the incident photon is chosen to lie in the x-z plane and it makes an angle α with the z-axis. The direction of the final photon has the angular coordinates (α', φ) as shown in Figure 2. In the coordinate system so chosen, we have

$$d\Omega_2 = d\varphi \, d\cos\alpha'$$
$$d\Omega_p = d\Omega_\nu = 2\pi \, d\cos\alpha \qquad . \qquad (16)$$

Then equation (15) becomes

$$\mu_s(u, kT, \eta) = \frac{8\pi}{\lambda_c^3} \int_{E'=0}^{\infty} dE' \int_{\cos\alpha=-1}^{1} d\cos\alpha \int_{\cos\alpha'=-1}^{1} d\cos\alpha' \int_{\varphi=0}^{\pi} d\varphi \, \frac{(E'^2+2E')^{1/2}(1+E')}{1+\exp(E'/T'-\eta)}$$
$$\left\{1 - \left[1+\exp(E'/T'+u-u_2-\eta)\right]^{-1}\right\} \frac{d\sigma}{d\Omega_2}\left(1 - \frac{\nu_2\cos\theta}{\nu}\right) \qquad . \qquad (17)$$

The scattering angle θ is easily shown to be given by

$$\cos\theta = \cos\alpha\cos\alpha' + \sin\alpha\sin\alpha'\cos\varphi \qquad . \qquad (18)$$

By the conservation laws ν_2 is related to ν and the other variables through

$$\frac{\nu_2}{\nu} = \frac{1-\beta\cos\alpha}{1-\beta\cos\alpha' + h\nu(1-\cos\theta)/E} \qquad , \qquad (19)$$

i.e.
$$\frac{u_2}{u} = \frac{1-\beta\cos\alpha}{1-\beta\cos\alpha' + u\,T'(1-\cos\theta)/(1+E')} \qquad . \qquad (20)$$

We use a form of the differential cross section given by Jauch and Rohlich (1955)

$$\frac{d\sigma}{d\Omega_2} = \frac{r_0^2}{2\gamma^2}\left(\frac{\nu_2}{\nu}\right)^2 \frac{X}{(1-\beta\cos\alpha)^2} \qquad , \qquad (21)$$

where r_o = classical radius of the electron and

$$X = \frac{\nu}{\nu_2}\,\frac{1-\beta\cos\alpha}{1-\beta\cos\alpha'} + \frac{1-\beta\cos\alpha'}{1-\beta\cos\alpha}\,\frac{\nu_2}{\nu} - 1$$
$$+ \left[1 + \frac{m^2c^4}{E\,h\nu(1-\beta\cos\alpha)} - \frac{m^2c^4}{E\,h\nu_2(1-\beta\cos\alpha')}\right]^2 \qquad . \qquad (22)$$

Thus, substituting equations (18) - (22) into equation (17), $\mu_s(u, kT, \eta)$ can be computed. The Rosseland mean is given by

$$\Lambda(kT, \eta) = \frac{15}{4\pi^4}\int_0^{\infty} \frac{1}{\mu_s(u, kT, \eta)}\,\frac{u^2 e^u}{(1-e^u)^2}\,du \qquad . \qquad (23)$$

We have neglected μ_a in the scattering-dominated region. In all, we have a 5-fold integration to do for each of the corresponding values of kT and η . The computation was done on an IBM 9094.

Defining $G(kT, \eta)$ as the transport cross section in units of the Thompson cross section σ_0 , then the Rosseland mean is

$$\Lambda(kT, \eta) = \frac{1}{\sigma_0 N G(kT, \eta)} \tag{24}$$

where N is the number of electrons given by equation (14). The opacity is $K = (\rho \Lambda)^{-1}$, where $(\sigma_0 N)^{-1}$ is the Rosseland mean in the classical limit. Values of $G(kT, \eta)$ are tabulated in Table 1 for the corresponding values of kT and η . The values corresponding to $\eta = -$"∞" were obtained from

$$\bar{G}(T) = -0.13887 + 4.9871(kT)^{-1/2} - 5.9479(kT)^{-1} - 2.362(kT)^{-3/2}$$
$$20\ \text{Kev} \le kT \le 125\ \text{Kev} \tag{25}$$

from Sampson (1959) for the nondegenerate case,

If the values of $G(kT, \eta)$ at other kT and η are needed, the following polynomial of kT and η will fit fairly well for the region under consideration with an error of approximately 5 %,

$$\log_e G(kT, \eta) = -0.3037 - 6.89757\frac{kT}{mc^2} + 8.89771\left(\frac{kT}{mc^2}\right)^2 - 0.158737\,\eta$$
$$+ 0.392553\frac{kT}{mc^2}\eta - 0.0146817\,\eta^2 - 0.451961\left(\frac{kT}{mc^2}\right)^2\eta - 0.0523759\frac{kT}{mc^2}\eta^2 \tag{26}$$

To facilitate application of the transport cross section due to Compton scattering, we list the number densities of the electron as given by equation (14) in Table 2.

DISCUSSION

We have so far neglected the positron contribution to Compton scattering. It can be shown that at $\eta = -1$ and $kT = .35\ mc^2$ which are most favorable to pair creation in the region under consideration the ratio of the number density of electrons and positrons to that of electrons is 1.027. At all other values of η and kT in Table 2 the ratio is less than 1.005. While the correction to the number density arising from the presence of positrons at $\eta = -1$ and $kT = .35\ mc^2$ is + 2.7 % the corresponding correction to effective cross section $\sigma(kT, \eta)$ should be much less than that since the latter is less sensitive to the presence of positrons than is the number density N.

The other correction due to positrons is from the absorption resulting in pair creation, this will not be significant until $kT \simeq 5\ mc^2$.

If our computation is repeated for $\eta = -3$, the results agree with Sampson's (1959) to within 2 %.

Sampson (1961) suggested that one estimate the effective degenerate cross section by multiplying his nondegenerate results by a factor

$$Q(T', \eta) = \left\{ \int_0^\infty \frac{(E+1)(E^2+2E)^{1/2}\, dE}{\left[1+\exp(E/T'-\eta)\right]\left[1+\exp(\eta-E/T')\right]} \right\} \left\{ \int_0^\infty \frac{(E+1)(E^2+2E)^{1/2}\, dE}{\left[1+\exp(E/T'-\eta)\right]} \right\}^{-1} \quad (27)$$

Upon comparison his estimates are smaller by approximately 10-20 % than ours in most cases.

ACKNOWLEDGEMENT

I would like to thank Professor Hong-Yee Chiu for suggesting the problem and Dr. Richard Stothers for discussions.

REFERENCES

Frank-Kamenetskii, D. 1962, Physical Processes in Stellar Interiors (Jerusalem: Israel Program for Scientific Translations).

Jauch, J. M. and Rohrlich, F. 1955, The Theory of Photons and Electrons (Reading, Massachusetts: Addison-Wesley).

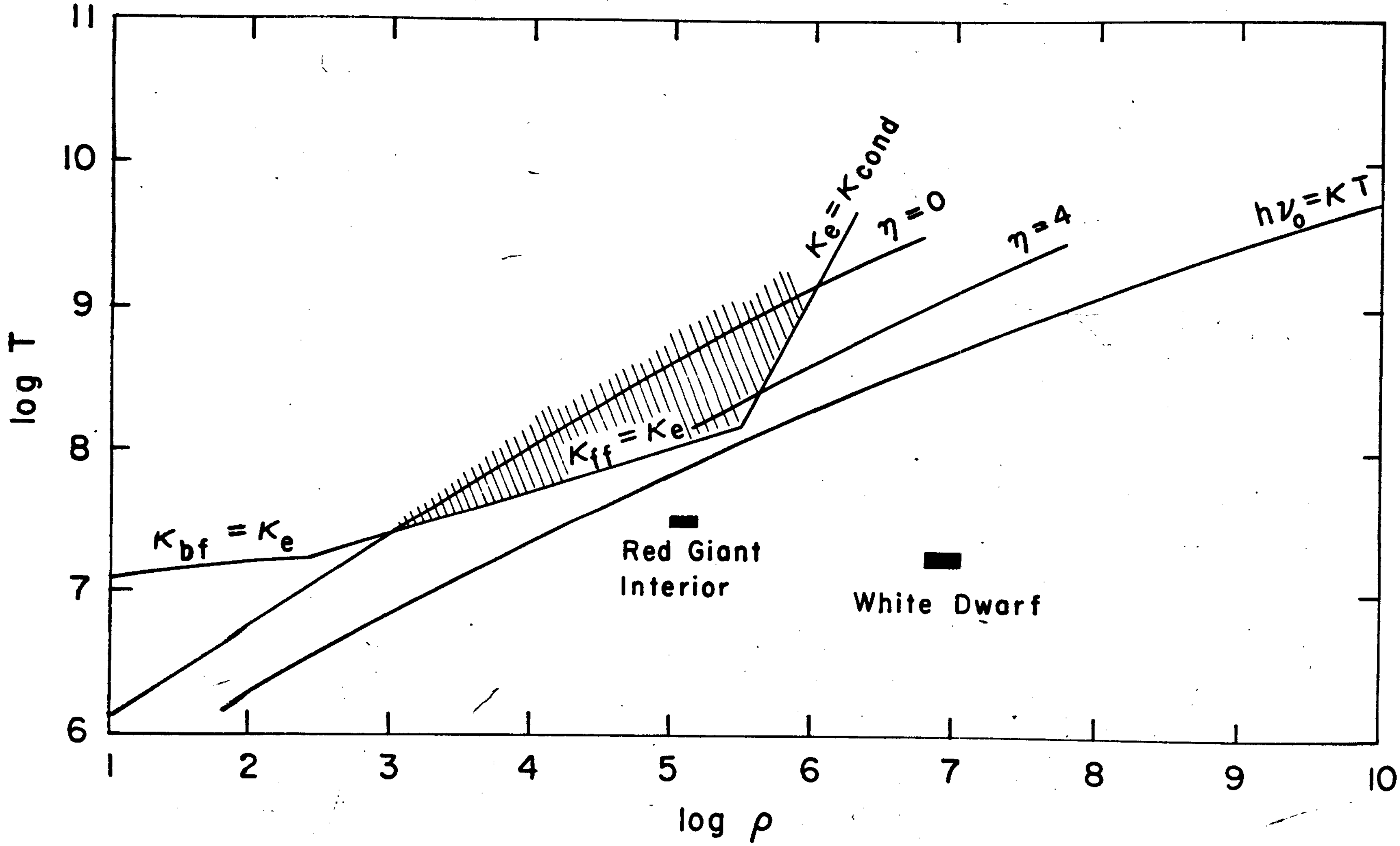
Sampson, D. H. 1959, Ap. J. **129**, 734.

__________ 1961, Ap. J. **134**, 482.

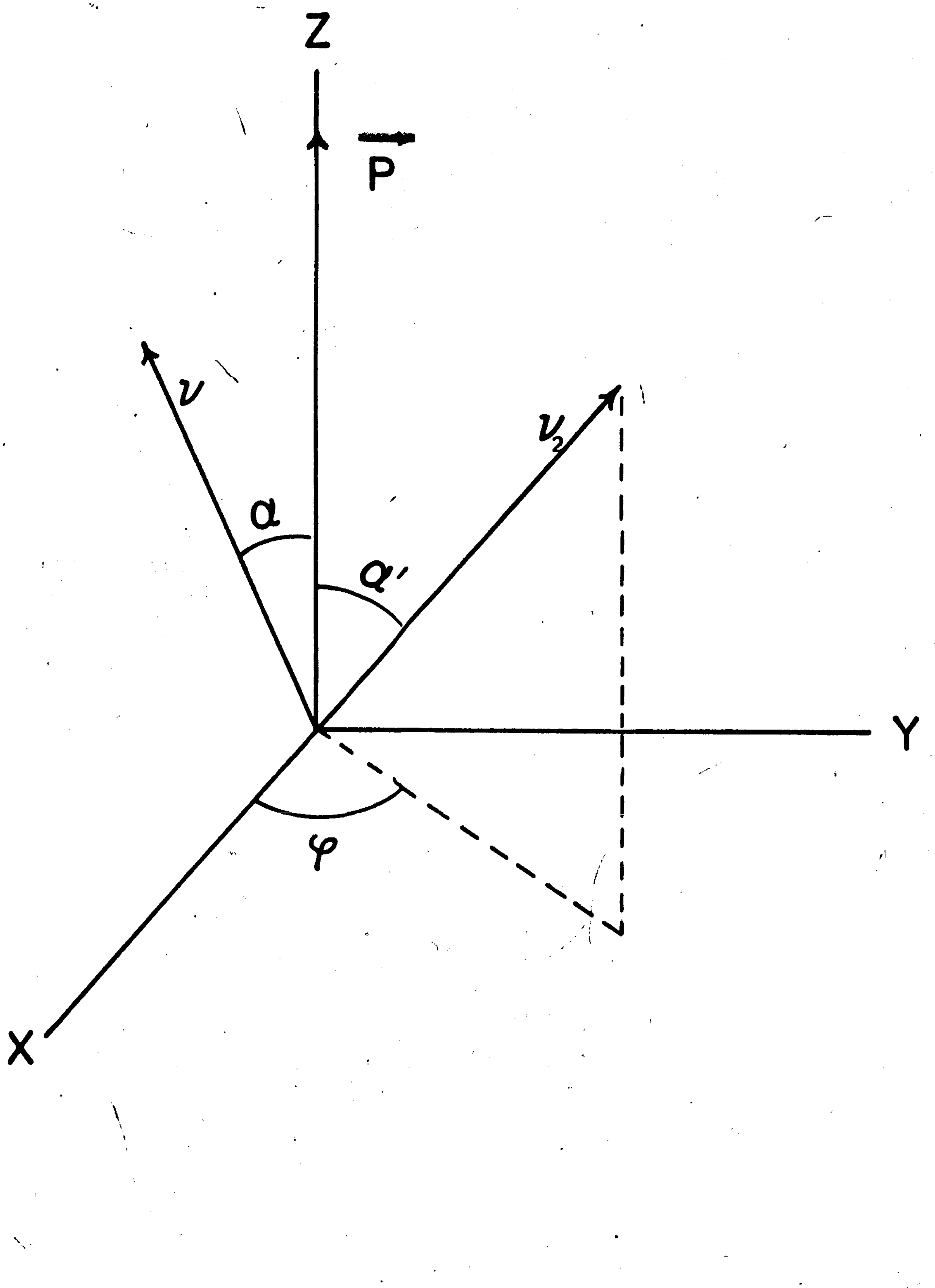
LEGENDS

Figure 1. - Temeprature-density diagram. The shaded area shows the region under consideration and ω_0 stands for plasma frequency.

Figure 2. - Coordinate system used in the calculation.



log T
11
10
9
8
7
6
1
2
3
4
5
6
7
8
9
10
log ρ
κbf = κe
κff = κe
κe = κcond
η = 0
η = 4
hν0 = KT
Red Giant
Interior
White Dwarf



Z
P
ν
ν₂
α
α'
Y
φ
X

TABLE 1. VALUES OF $G(kT, \eta)$ FOR VARIOUS TEMPERATURES AND DEGENERACY PARAMETERS η.

$kT/(mc^2)$	η					
	$-$"∞"	-1	0	1	2	4
.01		.8294	.7368	.6112	.4864	.3094
.03		.6732	.6060	.5094	.4076	.2554
.05	.5967	.5699	.5189	.4410	.3553	.2217
.15	.3496	.3416	.3194	.2816	.2322	.1423
.25	.2542	.2529	.2389	.2135	.1776	.1066
.35		.2028	.1923	.1730	.1443	.0843

TABLE 2. VALUES OF $N_e \lambda_c^3$ AT VARIOUS VALUES OF kT and η.

$kT/(mc^2)$	η				
	-1	0	+1	+2	+4
.01	0.01046	0.02448	0.05061	0.09077	0.2117
.03	0.05650	0.1326	0.2755	0.4974	1.183
.05	0.1264	0.2972	0.6195	1.129	2.727
.15	0.7847	1.865	3.963	7.442	19.25
.25	1.983	4.754	10.25	19.69	53.45
.35	3.804	9.180	20.03	39.14	110.2